Laurie H. Rubel

Is 7/10 Always Equivalent to 700/1000?

Consider the following problem:

> Compare the probabilities of the following two events: (1) 7 tails out of 10 tosses of a fair coin and (2) 700 tails out of 1000 tosses of a fair coin. Which of the events is more likely, or are they equally likely? (adapted from Fischbein and Schnarch 1997, p. 99)

I asked students this question as part of a study (Rubel 2002) that explored students' probabilistic thinking. I gathered written responses and justifications from 173 students in grades 5, 7, 9, and 11 and conducted interviews with 33 of those students. During the interviews, students referred to a solution that differed from and conflicted with their written answer. I will present and interpret students' responses and discuss their alternative approaches.

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Edited by ***Thomas A. Evitts,*** *TAEvit@ship.edu*
Shippensburg University, Shippensburg, PA 17257

Karen Heinz, *Heinz@rowan.edu*
Rowan University, Glassboro, NJ 08028

STUDENTS' RESPONSES TO THE COIN COMPARISON TASK

Nearly all students gave one of the following two answers to the written coin comparison task: (1) 7 tails out of 10 tosses is more likely (the correct answer) or (2) the two events are equally likely. **Table 1** depicts the distribution of responses.

Justification of the Correct Answer

Only 23 percent of the students gave the correct answer: 7 tails out of 10 tosses is more likely. The most common justification for the correct answer referred in some way to the effect of sample size. Using their own language, students stated a preliminary understanding of the law of large numbers—that we would expect the relative frequency of tails to be closer to 50 percent in the larger sample.

A second approach to the correct answer involved "exactness." For 10 tosses of a coin, there are 11 (unequally likely) options for the number of tails; there could be 0, 1, 2, 3, ..., or 10 tails. In contrast, for 1000 tosses of a coin, there are 1001 (again, unequally likely) options for the number of tails. Some students reasoned that because there are so many more options for the number of tails in the larger sample, landing on any specific one would be unlikely. This is true: Getting exactly 700 tails on 1000 tosses *is* highly unlikely. Such reasoning might help students understand why, for instance, exactly 15 tails in 30 tosses is much more likely than exactly 300 tails in 600 tosses. This exactness approach leads to the correct answer in the original question and in this example, in which the sample sizes are of decidedly different magnitudes, but it will not lead to a correct comparison in other instances.

Table 1 Distribution of Responses in Each Grade Level

	Grade 5 ($n = 36$)	**Grade 7** ($n = 45$)	**Grade 9** ($n = 50$)	**Grade 11** ($n = 42$)	**Total** ($n = 173$)
7 tails out of 10 tosses is more likely	17% (6)	13% (6)	22% (11)	40% (17)	23% (40)
Events are equally likely	78% (28)	87% (39)	68% (34)	55% (23)	72% (124)

Analysis of the Incorrect Answer

Seventy-two percent of the students answered, in writing, that 7 tails out of 10 tosses is as likely as 700 tails out of 1000 tosses. Why did so many students make this error? One explanation is that this is an example of Same A Same B thinking: If two systems or objects are equal in terms of a quality A, then those two objects will also be equal in terms of a second quality B (Tirosh and Stavy 2000). The most common student justification for the incorrect answer was that an equal ratio of tails to total coin tosses—7:10 = 700:1000—implies that the probabilities are equal.

A second explanation is that we can correctly solve other probability tasks by comparing ratios. For instance, in comparing Bag A (7 red and 3 blue marbles) with Bag B (700 red and 300 blue marbles), we can compare the ratio of red marbles to total marbles in each bag. Because the ratios are equal, we can conclude that we would be equally likely to pull a red marble out of Bag A as we would out of Bag B. In other words, in some probabilistic situations, Same A (ratios) implies Same B (probabilities).

A third explanation for why so many students stated that the two events were equally likely is that students could have drawn on only what they know from their school mathematics experience and not on what they know from their out-of-school mathematics experience. Excerpts from my interviews with two students illustrate this phenomenon.

MATHEMATICS IN AND OUT OF SCHOOL

One Seventh Grader's Thinking

On the written task, Kendall (a pseudonym), a seventh grader, wrote that 7 tails out of 10 tosses is as likely as 700 tails out of 1000 tosses. In his interview, he revealed additional ideas.

Kendall: I have to say that even though both reduce to the same fraction, it would be more likely that you'd get 7 out of 10 because then your chances—actually, well, if you're going more likely, probably 7 out of 10 because the higher you go up, the less chance you have of being that precise. You know what I'm saying?

Researcher: Not exactly.

Kendall: If I flipped a coin ten times, I'd think, again, not on paper, this is more real life, there's probably more of a chance than what you can do on paper. Ten times, there's more of a chance of getting 7. If you flip it 1000 times, getting 700 is, I mean it's possible. For some reason 7 out of 10 just jumps out at me.

Researcher: Were you thinking about this since last week?

Kendall: That's not what I wrote. I've been thinking about it, what would be more likely. My strategy here is on paper.

Researcher: The math way? [*Earlier in this interview, Kendall had mentioned "the math way."*]

Kendall: If someone just came up to me and asked me this, I would definitely say 7 out of 10 because 700 out of 1000 just seems like more.

Researcher: You're a basketball player, right? Do you ever take free throws and keep track of how many you make?

Kendall: Yes. The larger number you get, the larger the amount of error. With the free-throw analogy, it's not too hard to get 7 out of 10. In real life, it would be hard to keep up that, I want to say, ratio.

Later in the interview, Kendall was asked to compare the likelihood that the Yankees will win 8 out of 10 games with the likelihood that they will win 80 out of 100 games.

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Kendall: It's easier to get 8 out of 10 in real life. But if you reduce them, it's 8 out of 10. In real life, if I was betting, I would not bet on 800 out of 1000.

Researcher: What if it were a question on a math test?

Kendall: The way I answered it on this. I'd say if you put it into a fraction, it's 8 out of 10. On paper, it makes more sense doing it this way. Usually in math, there's one answer. 8 out of 10 is eight tenths. 800 out of 1000 is eight tenths, and I know that's correct. That's how I would answer on a math test.

These Kendall vignettes demonstrate the significance of a student's beliefs about what it means to do mathematics in or out of school. One interpretation of this contrast is that "school represents a specialized set of educational experiences which are discontinuous from those encountered in everyday life and … [school] requires and promotes ways of learning and thinking which often run counter to those nurtured in practical daily activities" (Scribner and Cole 1973, p. 553).

In Kendall's interview, he explained his "exactness" approach to thinking about the problem as well as ideas about the effect of sample size but characterized those methods as being appropriate only for outside the classroom. During the interview, I introduced the basketball context because I knew Kendall had experience with shooting sets of free throws. This context seemed to help him articulate the difference between the ratios and the respective probabilities. Yet in the face of these ways of thinking about the problem, Kendall maintained that the incorrect solution, the one that demonstrated that both probabilities are equal, was the appropriate solution in a mathematics classroom.

An Older Student's Thinking

By examining an excerpt from an interview with Ned, an eleventh grader, we gain some additional insight into this phenomenon.

Ned: The pure math is that they're the same thing. The probability of 7 out of 10 is the same as 70 out of 100. It's like 7 tenths. But if you actually do it 1000 times, it gets less likely to get 700.… On a long enough span, it's going to be harder to get this percentage.

Researcher: So, are you sticking with this answer?

Ned: There are two ways to look at it. I chose this way.

Researcher: How do you choose if there are two answers?

Ned: The way where you say divide 7 by 10 or 700 by 1000. It's always gonna be this—there's no gray. It's less easy to prove the other way, even though I understand that there's another way to look at this.

Researcher: Let me change the question. How about comparing getting 10 tails out of 10 tosses with [getting] 1000 tails out of 1000 tosses?

Ned: (*pausing*) That's more like the other way.

Researcher: What about your way: 10 out of 10 is the same as 1000 out of 1000.

Ned: Obviously, they're not. Obviously, it's not the same: 1 out of 1 or 1000 out of 1000. So from thinking about it, it's not the same possibility. But on paper, it is.… OK, so 700 out of 1000 tails and 7 out of 10 tails. How much more likely is 7 out of 10?

Researcher: You said they were equally likely.

Ned: I know, on paper, they're equally likely. But if you're going to say one is more likely, then how much more likely? I want to know how much more. If you can't figure it out, then the way you can figure it out is the way I did it.

As Kendall did, Ned made connections to a mathematics experience outside school. At the same time, he positioned the erroneous Same A Same B approach as being "pure math," and, as Kendall did, he stated that the correct way to proceed "on paper" was to say that the probabilities were equal because the ratios were equal. It is provocative that both students accepted two conflicting answers to the same question. During the interview, when I presented Ned with task modifications in an attempt to reconcile this conflict, he persisted that "on paper" the two events are equally likely. He did not believe that it was possible to quantify the likelihood of either frequency. For Ned, quantification seemed to be a necessary component of correct thinking "on paper" (i.e., in school).

IMPLICATIONS

The results described here have several implications. The National Council of Teachers of Mathematics (NCTM) reminds us that students should "become knowledgeable, analytical, thoughtful consumers of the information and data generated by others" (NCTM 2000, p. 325). Achieving this goal requires an understanding of the effect of sample size in a random experiment, a fundamental concept in probability and statistics.

The coin comparison task could be a fruitful way to help students develop an understanding of the effect of sample size or to introduce the concept of the law of large numbers. Teachers can use findings from this study to anticipate how students might reason about such tasks. Instruction could be organized to confront the Same A Same B issue directly by altering the question to compare, for instance, 9 tails out of 10 tosses with 9,000,000 tails out of 10,000,000 tosses.

Classroom experiments to demonstrate the effects of sample size or the law of large numbers have a caveat. A typical setup has students perform reiterations of an experiment (tossing coins, spinning spinners, drawing from a bag, or conducting computer simulations of these) with a probability of success, p. Then, the teacher aggregates the individual data sets, and the new data set is assumed to show that the relative frequency of success is close to the probability of success, p. However, as many teachers may have experienced, it is not always obvious just how many trials are necessary to be sure that this convergence occurs.

In a classroom setting, how can we experientially address the concept of "large" in the law of large numbers? The law of large numbers describes results of an infinite process. When we try to model it with a finite process, even one that is quite large, the result might not match the predicted results, which assume infinitely many trials. If an experiment produces a relative frequency of success that is not close enough to the theoretical probability of success, this outcome can be misleading.

This study also highlights a limitation of assessing students only by evaluating written responses. The interview excerpts demonstrate that students can have conflicting responses to a given task, but this multiplicity was not evident in their written responses. On paper, it seemed that these students all had the common Same A Same B misconception, yet, when interviewed, they revealed that they also had other ways of thinking.

Once I noticed that students were differentiating between a "math answer" and a "real world answer," I adapted my interview protocol. After students explained their written answer, I asked them whether they had another approach to the problem that they would use if they were outside school. When working on problems with real-world contexts, related to probability or to other areas of mathematics, teachers might find it informative and useful to ask students how they might solve the problem outside school.

Finally, why students might view a distinction between thinking in the "real world" and thinking in a mathematics classroom is of utmost importance. We typically categorize school mathematics tasks as being either context free or related to the "real world." But because the "real world" mathematics tasks that we pose in the classroom are typically stripped of most of their contextual features (see Boaler 2008 for a discussion), perhaps we have inadvertently given students a message that problem solving in mathematics classrooms is different from problem solving outside school. This study provides evidence that an important component of developing students' mathematical understanding is to help them coordinate their knowledge about real-life situations with mathematics.

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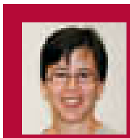

LAURIE H. RUBEL, LRubel@brooklyn.cuny.edu, is an associate professor of education at Brooklyn College of the City University of New York. She especially enjoys working with and learning from New York City high school teachers.

WILDA GALLAGHER